

Algebraic K -theory I and II

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- ▶ The abelian group of integers $\mathbb{Z}$ can be thought of as the group which is obtained from the commutative monoid $\mathbb{N}_{\geq 0}$ of the non-negative integers by formally adding inverses. This fundamental procedure of defining negative integers can be generalized to a much broader context.

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- ▶ The abelian group of integers $\mathbb{Z}$ can be thought of as the group which is obtained from the commutative monoid $\mathbb{N}_{\geq 0}$ of the non-negative integers by formally adding inverses. This fundamental procedure of defining negative integers can be generalized to a much broader context.
- ▶ Given a commutative monoid $(M, +)$, one may ask whether there exists a best possible approximation of $(M, +)$ by an abelian group. This can be indeed achieved by formally adding inverses to $(M, +)$ while also keeping the relations of $(M, +)$. The abelian group obtained in this way, denoted by $G(M, +)$, is called the *Grothendieck group* or the *group completion* of $(M, +)$.

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- More formally, the group $G(M, +)$ is defined by

$$G(M, +) = \mathbb{Z}[M] / \langle [a + b] - [a] - [b] \mid a, b \in M \rangle,$$

where $\mathbb{Z}[M]$ is the free abelian group generated by the set M and the symbol $[a]$ denotes the basis element corresponding to $a \in M$.

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where $\mathbb{Z}[M]$ is the free abelian group generated by the set M and the symbol $[a]$ denotes the basis element corresponding to $a \in M$.

- ▶ The group $G(M, +)$ is uniquely determined up to a unique isomorphism by the following universal property: Given a monoid homomorphism $f: (M, +) \rightarrow (H, +)$, where $(H, +)$ is an abelian group, there exists a unique group homomorphism $\bar{f}: G(M, +) \rightarrow (H, +)$ extending f . In particular, we have $G(\mathbb{N}_{\geq 0}, +) = (\mathbb{Z}, +)$ which can be taken as the formal definition of the group of integers.

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- ▶ Define: $I(k) = \text{Vect}_k / \sim$, where $V \sim W$ if and only if they are isomorphic.
- ▶ $I(k)$ is a commutative monoid with $[V] + [W] = [V \oplus W]$.
- ▶ $K_0(k) = G(I(k), +)$ is the 0-th algebraic K -group of k .
- ▶ In fact $\dim$ induces isomorphisms $I(k) \cong \mathbb{N}_{\geq 0}$ and $K_0(k) \cong \mathbb{Z}$ recovering our first example.

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- ▶ These are matrices of the form $e_{ij}(\lambda) = I + \lambda E_{ij}$, where $i \neq j$, I is the identity matrix, $\lambda \in k$, and E_{ij} is the matrix with 1 in the (i, j) entry, and zeros everywhere else.

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- ▶ Relations $e_{ij}(\lambda)e_{ij}(\mu) = e_{ij}(\lambda + \mu)$, $[e_{ij}(\lambda), e_{kl}(\mu)] = 1$, if $j \neq k$ and $i \neq l$, $[e_{ij}(\lambda), e_{jl}(\mu)] = e_{il}(\lambda\mu)$, if $i \neq l$.

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- ▶ Steinberg group $St(k)$ is the universal group with $e_{ij}(\lambda)$ as generators and above relations.

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- ▶ $K_2(k)$ was created by measuring how much information we lost by dividing out the commutator subgroup.
- ▶ Now we have K_0 , K_1 and K_2 . And the story goes on!

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- ▶ The groups $K_i(R)$ will see what kind of higher information is hidden in linear algebra over R .
- ▶ Algebraic K -theory is a very fancy linear algebra, or homotopy theory meets linear algebra. Fancy version of the group completion.

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- ▶ Literature: Any book on algebraic K -theory. I obviously will advertise:
- ▶ I. Madsen and I. Patchkoria: Introduction to Algebraic K -Theory, Compact textbooks in Maths, Birkhäuser.

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- ▶ Semester II: The K -theory spaces, additivity, cofinality, resolution theorem, Devissage and localization, computations.

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- ▶ Analysis: X a topological space: $C(X)$ the ring of complex valued of continuous functions. Then $K_0(C(X)) = K^0(X)$, the vector bundle topological K -theory. The group $K^0(X)$ shows up in the index theory of elliptic operators.

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- ▶ Number theory: If R is a Dedekind domain, then $K_0(R) = \mathbb{Z} \oplus Cl(R)$, where $Cl(R)$ is the ideal class group.
- ▶ $K_n(\mathbb{Z})$ not all computed but for example $|K_{4m-2}(\mathbb{Z})|$ "is basically" numerator of $\zeta(1 - 2m)$.

Thanks!